Appendix A

Transformed model equations

A.1 Horizontal transformation

The continuity, momentum and scalar transport equations can be rewritten from Cartesian to orthogonal curvilinear coordinates (ξ_1,ξ_2) with the aid of the general tranfsormation rules (e.g. Batchelor, 1979):

$$\nabla_h = \left(\frac{1}{h_1}\frac{\partial}{\partial\xi_1}, \frac{1}{h_2}\frac{\partial}{\partial\xi_2}\right) \tag{A.1}$$

$$\nabla_h \cdot \mathbf{F}_h = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial\xi_1}(h_2 F_1) + \frac{\partial}{\partial\xi_2}(h_1 F_2)\right] \tag{A.2}$$

$$\begin{aligned}\mathbf{F}_h \cdot \nabla_h \mathbf{F}_h = \Bigg[\mathbf{F}_h \cdot \nabla F_1 \quad &+ \quad \frac{F_2}{h_1 h_2}\left(F_1\frac{\partial h_1}{\partial\xi_2} - F_2\frac{\partial h_2}{\partial\xi_1}\right), \\ \mathbf{F}_h \cdot \nabla F_2 \quad &+ \quad \frac{F_1}{h_1 h_2}\left(F_2\frac{\partial h_2}{\partial\xi_1} - F_1\frac{\partial h_1}{\partial\xi_2}\right)\Bigg]\end{aligned} \tag{A.3}$$

where the subscript h denotes the horizontal component of the associated vector or operator and h_1, h_2 are the metric coefficients defined by (4.7). Substituting the above relations into (4.43)–(4.45) and (4.47) or (4.48) one obtains

$$\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial\xi_1}(h_2 u) + \frac{\partial}{\partial\xi_2}(h_1 v)\right] + \frac{\partial w}{\partial z} = 0 \tag{A.4}$$

$$\begin{aligned}&\frac{\partial u}{\partial t} + \frac{u}{h_1}\frac{\partial u}{\partial\xi_1} + \frac{v}{h_2}\frac{\partial u}{\partial\xi_2} + w\frac{\partial u}{\partial z} + \frac{v}{h_1 h_2}\left(u\frac{\partial h_1}{\partial\xi_2} - v\frac{\partial h_2}{\partial\xi_1}\right) - 2\Omega v\sin\phi \\ &\quad = -\frac{g}{h_1}\frac{\partial\zeta}{\partial\xi_1} - \frac{1}{\rho_o h_1}\frac{\partial P_a}{\partial\xi_1} - \frac{1}{h_1}\frac{\partial q}{\partial\xi_1} + F_1^t + \frac{\partial}{\partial z}\left(\nu_T\frac{\partial u}{\partial z}\right)\end{aligned}$$

$$+\mathcal{D}^*_{mh1}(\tau_{11}) + \mathcal{D}^*_{mh2}(\tau_{12}) \tag{A.5}$$

$$\begin{aligned}\frac{\partial v}{\partial t} + \frac{u}{h_1}\frac{\partial v}{\partial \xi_1} + \frac{v}{h_2}\frac{\partial v}{\partial \xi_2} + w\frac{\partial v}{\partial z} + \frac{u}{h_1 h_2}\left(v\frac{\partial h_2}{\partial \xi_1} - u\frac{\partial h_1}{\partial \xi_2}\right) + 2\Omega u \sin\phi \\ = -\frac{g}{h_1}\frac{\partial \zeta}{\partial \xi_2} - \frac{1}{\rho_o h_2}\frac{\partial P_a}{\partial \xi_2} - \frac{1}{h_2}\frac{\partial q}{\partial \xi_2} + F_2^t + \frac{\partial}{\partial z}\left(\nu_T \frac{\partial v}{\partial z}\right) \\ +\mathcal{D}^*_{mh1}(\tau_{21}) + \mathcal{D}^*_{mh2}(\tau_{22})\end{aligned} \tag{A.6}$$

$$\begin{aligned}\frac{\partial \psi}{\partial t} &+ \frac{u}{h_1}\frac{\partial \psi}{\partial \xi_1} + \frac{v}{h_2}\frac{\partial \psi}{\partial \xi_2} + w\frac{\partial \psi}{\partial z} = \mathcal{S}(\psi) + \frac{\partial}{\partial z}\left(\lambda_T \frac{\partial \psi}{\partial z}\right) \\ &+ \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \xi_1}\left(\lambda_H \frac{h_2}{h_1}\frac{\partial \psi}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\lambda_H \frac{h_1}{h_2}\frac{\partial \psi}{\partial \xi_2}\right)\right]\end{aligned} \tag{A.7}$$

The horizontal diffusion operators for momentum are defined by (Pacanowski & Griffies, 2000)

$$\mathcal{D}^*_{mh1}(F) = \frac{1}{h_1 h_2^2}\frac{\partial}{\partial \xi_1}(h_2^2 F) \tag{A.8}$$

$$\mathcal{D}^*_{mh2}(F) = \frac{1}{h_1^2 h_2}\frac{\partial}{\partial \xi_2}(h_1^2 F) \tag{A.9}$$

A.2 Vertical transformation

A general vertical coordinate is defined through the transformation

$$(\xi_1, \xi_2, z, t) \longrightarrow (\tilde{\xi}_1, \tilde{\xi}_2, s, \tilde{t}) \tag{A.10}$$

with $\tilde{\xi}_i = \xi_i$, $\tilde{t} = t$ and $s = f(\xi_1, \xi_2, z, t)$ where, as stated in Section 4.1.4.3, the transformed vertical coordinate s is defined by normalising the σ-coordinate, using (4.40) such that (A.13) is valid. Spatial and time derivatives are transformed by applying the chain rule

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}} + \frac{\partial s}{\partial t}\frac{\partial}{\partial s} \tag{A.11}$$

$$\frac{\partial}{\partial \xi_i} = \frac{\partial}{\partial \tilde{\xi}_i} + \frac{\partial s}{\partial \xi_i}\frac{\partial}{\partial s} \tag{A.12}$$

$$\frac{\partial}{\partial z} = \frac{1}{h_3}\frac{\partial}{\partial s} \tag{A.13}$$

where the derivatives on the left hand side in the first two relations are taken along constant z-surfaces and the first ones on the right side along constant s-surfaces. The following useful relations can be derived from (A.11)–(A.13)

$$\frac{\partial s}{\partial z} = \frac{1}{h_3}, \quad \frac{\partial z}{\partial s} = h_3 \tag{A.14}$$

$$\frac{\partial z}{\partial \tilde{\xi}_i} + h_3 \frac{\partial s}{\partial \xi_i} = 0 \tag{A.15}$$

$$\frac{\partial h_3}{\partial \tilde{t}} + \frac{\partial s}{\partial t}\frac{\partial h_3}{\partial s} = 0 \tag{A.16}$$

$$\frac{\partial h_3}{\partial \tilde{\xi}_i} = \frac{\partial}{\partial \tilde{\xi}_i}\left(\frac{\partial z}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial \tilde{\xi}_i}\right) = -\frac{\partial}{\partial s}\left(h_3 \frac{\partial s}{\partial \xi_i}\right) \tag{A.17}$$

$$\frac{\partial z}{\partial \tilde{t}} = -\frac{\partial s}{\partial t}\frac{\partial z}{\partial s} = -h_3 \frac{\partial s}{\partial t} \tag{A.18}$$

A new vertical velocity is defined by

$$\begin{aligned}
\omega &= h_3 \frac{ds}{dt} \\
&= h_3 \left(\frac{\partial s}{\partial t} + \frac{u}{h_1}\frac{\partial s}{\partial \xi_1} + \frac{v}{h_2}\frac{\partial s}{\partial \xi_2} + w\frac{\partial s}{\partial z}\right) \\
&= h_3 \left(\frac{\partial s}{\partial t} + \frac{u}{h_1}\frac{\partial s}{\partial \xi_1} + \frac{v}{h_2}\frac{\partial s}{\partial \xi_2}\right) + w
\end{aligned} \tag{A.19}$$

from which (4.72) is obtained.

The continuity equation (A.4) is rewritten in the transformed coordinate system with the aid of the previous relations

$$\begin{aligned}
0 &= \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \xi_1}(h_2 u) + \frac{\partial}{\partial \xi_2}(h_1 v)\right] + \frac{\partial w}{\partial z} \\
&= \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \tilde{\xi}_1}(h_2 u) + \frac{\partial}{\partial \tilde{\xi}_2}(h_1 v) + \frac{\partial s}{\partial \xi_1}\frac{\partial}{\partial s}(u h_2) + \frac{\partial s}{\partial \xi_2}\frac{\partial}{\partial s}(v h_1)\right] + \frac{1}{h_3}\frac{\partial w}{\partial s} \\
&= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \tilde{\xi}_1}(h_2 h_3 u) + \frac{\partial}{\partial \tilde{\xi}_2}(h_1 h_3 v) + h_3 \frac{\partial s}{\partial \xi_1}\frac{\partial}{\partial s}(u h_2) - u h_2 \frac{\partial h_3}{\partial \tilde{\xi}_1}\right. \\
&\quad \left. + h_3 \frac{\partial s}{\partial \xi_2}\frac{\partial}{\partial s}(v h_1) - v h_1 \frac{\partial h_3}{\partial \tilde{\xi}_2}\right] + \frac{1}{h_3}\frac{\partial w}{\partial s} \\
&= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \tilde{\xi}_1}(h_2 h_3 u) + \frac{\partial}{\partial \tilde{\xi}_2}(h_1 h_3 v) + h_3 \frac{\partial s}{\partial \xi_1}\frac{\partial}{\partial s}(u h_2) + u h_2 \frac{\partial}{\partial s}\left(h_3 \frac{\partial s}{\partial \xi_1}\right)\right. \\
&\quad \left. + h_3 \frac{\partial s}{\partial \xi_2}\frac{\partial}{\partial s}(v h_1) + v h_1 \frac{\partial}{\partial s}\left(h_3 \frac{\partial s}{\partial \xi_2}\right)\right] + \frac{1}{h_3}\frac{\partial w}{\partial s}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\tilde{\xi}_1}(h_2h_3u) + \frac{\partial}{\partial\tilde{\xi}_2}(h_1h_3v) + \frac{\partial}{\partial s}\left(h_2h_3u\frac{\partial s}{\partial\xi_1} + h_1h_3v\frac{\partial s}{\partial\xi_2}\right)\right] + \frac{1}{h_3}\frac{\partial w}{\partial s} \\
&= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\tilde{\xi}_1}(h_2h_3u) + \frac{\partial}{\partial\tilde{\xi}_2}(h_1h_3v)\right] + \frac{1}{h_3}\frac{\partial}{\partial s}\left[w + \frac{h_3}{h_1}u\frac{\partial s}{\partial\xi_1} + \frac{h_3}{h_2}v\frac{\partial s}{\partial\xi_2}\right] \\
&= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\tilde{\xi}_1}(h_2h_3u) + \frac{\partial}{\partial\tilde{\xi}_2}(h_1h_3v)\right] + \frac{1}{h_3}\frac{\partial}{\partial s}\left(\omega - h_3\frac{\partial s}{\partial t}\right) \\
&= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\tilde{\xi}_1}(h_2h_3u) + \frac{\partial}{\partial\tilde{\xi}_2}(h_1h_3v)\right] + \frac{1}{h_3}\frac{\partial\omega}{\partial s} - \frac{1}{h_3}\frac{\partial h_3}{\partial s}\frac{\partial s}{\partial t} - \frac{\partial}{\partial s}\left(\frac{\partial s}{\partial t}\right) \\
&= \frac{1}{h_3}\frac{\partial h_3}{\partial\tilde{t}} + \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\tilde{\xi}_1}(h_2h_3u) + \frac{\partial}{\partial\tilde{\xi}_2}(h_1h_3v)\right] + \frac{1}{h_3}\frac{\partial\omega}{\partial s} \qquad \text{(A.20)}
\end{aligned}
$$

which becomes identical to (4.60) by letting $\tilde{\xi}_i = \xi_i$ and $\tilde{t} = t$.

The physical vertical current is given by

$$
\begin{aligned}
w &= \frac{dz}{dt} = \frac{\partial z}{\partial\tilde{t}} + \frac{u}{h_1}\frac{\partial z}{\partial\tilde{\xi}_1} + \frac{v}{h_2}\frac{\partial z}{\partial\tilde{\xi}_2} + \frac{\omega}{h_3}\frac{\partial z}{\partial s} \\
&= \frac{\partial z}{\partial\tilde{t}} + \frac{u}{h_1}\frac{\partial z}{\partial\tilde{\xi}_1} + \frac{v}{h_2}\frac{\partial z}{\partial\tilde{\xi}_2} + \omega \qquad \text{(A.21)}
\end{aligned}
$$

Equation (4.73) is recovered by adding (A.21) and z times (A.20)

$$
\begin{aligned}
w &= \frac{\partial z}{\partial\tilde{t}} + \frac{u}{h_1}\frac{\partial z}{\partial\tilde{\xi}_1} + \frac{v}{h_2}\frac{\partial z}{\partial\tilde{\xi}_2} + \omega \\
&\quad + \frac{z}{h_3}\left[\frac{1}{h_1h_2}\left(\frac{\partial}{\partial\tilde{\xi}_1}(h_2h_3u) + \frac{\partial}{\partial\tilde{\xi}_2}(h_1h_3v)\right) + \frac{\partial\omega}{\partial s} + \frac{\partial h_3}{\partial\tilde{t}}\right] \\
&= \frac{1}{h_3}\left[\omega\frac{\partial z}{\partial s} + z\frac{\partial\omega}{\partial s} + h_3\frac{\partial z}{\partial\tilde{t}} + z\frac{\partial h_3}{\partial\tilde{t}} + \frac{uh_3}{h_1}\frac{\partial z}{\partial\tilde{\xi}_1} + \frac{z}{h_1h_2}\frac{\partial}{\partial\tilde{\xi}_1}(h_2h_3u)\right. \\
&\quad \left. + \frac{vh_3}{h_2}\frac{\partial z}{\partial\tilde{\xi}_2} + \frac{z}{h_1h_2}\frac{\partial}{\partial\tilde{\xi}_2}(h_1h_3v)\right] \\
&= \frac{1}{h_3}\left[\frac{\partial}{\partial\tilde{t}}(h_3z) + \frac{1}{h_1h_2}\frac{\partial}{\partial\tilde{\xi}_1}(h_2h_3uz) + \frac{1}{h_1h_2}\frac{\partial}{\partial\tilde{\xi}_2}(h_1h_3vz) + \frac{\partial}{\partial s}(\omega z)\right] \\
&\qquad \text{(A.22)}
\end{aligned}
$$

The total derivative of a quantity ψ (velocity component or scalar) transforms according to

$$
\begin{aligned}
\frac{d\psi}{dt} &= \frac{\partial\psi}{\partial t} + \frac{u}{h_1}\frac{\partial\psi}{\partial\xi_1} + \frac{v}{h_2}\frac{\partial\psi}{\partial\xi_2} + w\frac{\partial\psi}{\partial z} \\
&= \frac{\partial\psi}{\partial\tilde{t}} + \frac{u}{h_1}\frac{\partial\psi}{\partial\tilde{\xi}_1} + \frac{v}{h_2}\frac{\partial\psi}{\partial\tilde{\xi}_2} + \frac{\partial\psi}{\partial s}\left(\frac{\partial s}{\partial t} + \frac{u}{h_1}\frac{\partial s}{\partial\xi_1} + \frac{v}{h_2}\frac{\partial s}{\partial\xi_2} + \frac{w}{h_3}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\partial\psi}{\partial\tilde{t}} + \frac{u}{h_1}\frac{\partial\psi}{\partial\tilde{\xi}_1} + \frac{v}{h_2}\frac{\partial\psi}{\partial\tilde{\xi}_2} + \frac{\omega}{h_3}\frac{\partial\psi}{\partial s} \\
&= \frac{1}{h_3}\frac{\partial}{\partial\tilde{t}}(h_3\psi) + \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi_1}(h_2h_3u\psi) + \frac{\partial}{\partial\xi_2}(h_1h_3v\psi)\right] + \frac{1}{h_3}\frac{\partial}{\partial s}(\psi\omega) \\
&\quad - \frac{\psi}{h_3}\frac{\partial h_3}{\partial\tilde{t}} - \frac{\psi}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi_1}(h_2h_3u) + \frac{\partial}{\partial\xi_2}(h_1h_3v)\right] - \frac{\psi}{h_3}\frac{\partial\omega}{\partial s} \\
&= \frac{1}{h_3}\frac{\partial}{\partial\tilde{t}}(h_3\psi) + \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi_1}(h_2h_3u\psi) + \frac{\partial}{\partial\xi_2}(h_1h_3v\psi)\right] + \frac{1}{h_3}\frac{\partial}{\partial s}(\psi\omega)
\end{aligned} \tag{A.23}
$$

by virtue of (A.20).

The horizontal gradient of a vertically independent quantity obviously does not change. For a 3-D quantity one has

$$
\begin{aligned}
\frac{1}{h_i}\frac{\partial\psi}{\partial\xi_i} &= \frac{1}{h_i}\left[\frac{\partial\psi}{\partial\tilde{\xi}_i} + \frac{\partial s}{\partial\xi_i}\frac{\partial\psi}{\partial s}\right] \\
&= \frac{1}{h_i}\left[\frac{1}{h_3}\frac{\partial}{\partial\tilde{\xi}_i}(h_3\psi) - \frac{\psi}{h_3}\frac{\partial h_3}{\partial\tilde{\xi}_i} + \frac{\partial s}{\partial\xi_i}\frac{\partial\psi}{\partial s}\right] \\
&= \frac{1}{h_i}\left[\frac{1}{h_3}\frac{\partial}{\partial\tilde{\xi}_i}(h_3\psi) + \frac{\psi}{h_3}\frac{\partial}{\partial s}\left(h_3\frac{\partial s}{\partial\xi_i}\right) + \frac{\partial s}{\partial\xi_i}\frac{\partial\psi}{\partial s}\right] \\
&= \frac{1}{h_i}\left[\frac{1}{h_3}\frac{\partial}{\partial\tilde{\xi}_i}(h_3\psi) + \frac{1}{h_3}\frac{\partial}{\partial s}\left(h_3\psi\frac{\partial s}{\partial\xi_i}\right)\right] \\
&= \frac{1}{h_ih_3}\left[\frac{\partial}{\partial\tilde{\xi}_i}(h_3\psi) - \frac{\partial}{\partial s}\left(\psi\frac{\partial z}{\partial\tilde{\xi}_i}\right)\right]
\end{aligned} \tag{A.24}
$$

from which (4.74) is obtained with $\psi = q$.

Applying the previous rule for the horizontal diffusion terms in the momentum and scalar transport equations one recovers the definitions (4.67), (4.68), (4.77) and (4.78) by making the assumption that diffusion takes place along constant s-surfaces.